# Bubble-and-bust dynamics under walrasian asset pricing and heterogeneous traders

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Pisa, July 11th, 2014

# Outline

1. Empirical stylised facts of financial time-series
2. Methodological perspective
3. The model
4. Simulation and results
5. Concluding remarks and conceivable extension

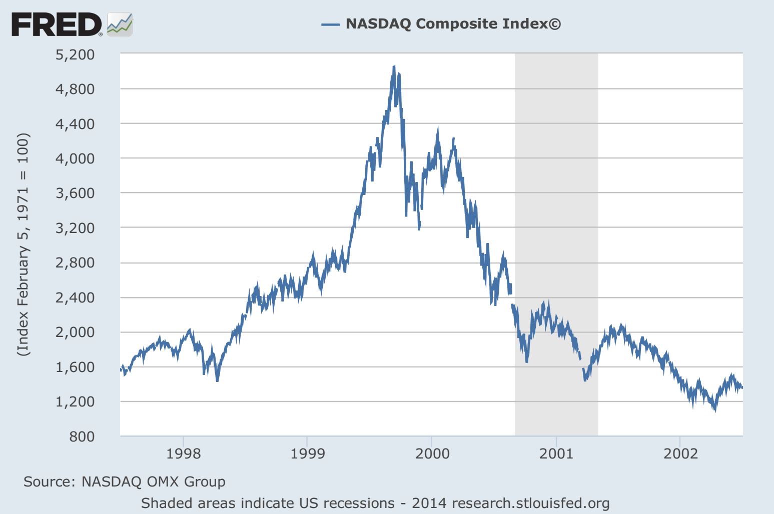
# Empirical stylised facts

Empirical analysis of actual financial data often yields puzzling statistical properties. Notably, asset returns time-series use to exhibit:

- Volatility clustering;
- Excess volatility;
- Excess covariance;
- Heavy tails.

To some extent at odds with accepted theory and practice.

# Financial bubbles: a largely neglected stylised fact



# Financial bubbles (cont'd)

*"if the reason that the price is high today is only because investors believe that the selling price will be high tomorrow – when 'fundamental' factors do not seem to justify such a price – then a bubble exists"* (Stiglitz)

*"a sharp rise in the price of an asset or a range of assets in a continuous process, with the initial rise generating expectations of further rises and attracting new buyers – generally speculators interested in profits from trading in the asset rather than its use or earnings capacity"* (Kindleberger)

# Financial bubbles (cont'd)

**Our proposal**

We set up a model able to yield:

- *endogenous* bubble-and-bust dynamics
- as a result of the sole interaction among *heterogeneous adaptive traders*
- highlighting booms and crashes as intrinsic features of financial markets.

# Methodological perspective

## Heterogeneous Agents Models

- analytical investigations of the dynamical systems representing the laws of motion of the economy;
- analytical tractability often leads to restrictive simplifying assumptions;
- focus on asymptotic properties.

## Agent-Based Models

- computational (numerical) study of economies modelled as evolving systems of interacting agents;
- complex behaviour specifications;
- keep track of the whole dynamics.

# The model

Following Anufriev et al. (2012), consider a pure-exchange economy:

- $N$ heterogeneous traders (index $\mathscr{N} = \{1, \ldots, n, \ldots, N\}$);
- $L$ long-lived risky securities (index $\mathscr{L} = \{1, \ldots, \ell, \ldots, L\}$);
- a riskless bond;
- time is discrete;
- risky securities, present in fixed amount, have ex-dividend price $p_t^\ell$ and pay a random dividend $d_t^\ell$ at the end of each period;
- the bond, inelastically supplied, have price normalized to 1 (*numéraire*) and yields $r_f > 0$ in every $t$;
- trader wealth equals the market value of the portfolio he holds:

$$W_{n,t} = \sum_{\ell=1}^{L} A_{n,t}^\ell \cdot p_t^\ell + B_{n,t}$$

# Trader behaviour

At the beginning of each time step, trader $n$ invests a share $x^{\ell}_{n,t}$ of his wealth in security $\ell$; the decision is made according to the information set

$$\mathcal{I}_t = \{p^1_\tau, \dots, p^L_\tau;\ d^1_\tau, \dots, d^L_\tau \mid \tau < t\}$$

that is common knowledge, and to trader-specific investment function

$$f_n : \mathbb{R}^{\tau \times L} \longrightarrow \mathbb{R}^L \quad \text{such that} \quad \mathbf{x}_{n,t} = f_{n,t}(\mathcal{I}_t)$$

that is independent on wealth, as we assume overall CRRA attitude. The amount of wealth invested in the bond is residually determined:

$$x^0_{n,t} = 1 - \sum_{\ell=1}^{L} x^{\ell}_{n,t}$$

The evolution of individual wealth, in terms of wealth fractions, holds:

$$W_{n,t} = W_{n,t-1} \cdot \left[ x^0_{n,t-1} \cdot (1 + r_f) + \sum_{\ell=1}^{L} x^{\ell}_{n,t-1} \cdot \left( \frac{p^{\ell}_t}{p^{\ell}_{t-1}} + e^{\ell}_t \right) \right]$$

# Useful additional definitions

**Dividend yield**: the ratio of dividend over past realized price (proxy for fundamentals)

$$e_t^\ell = \frac{d_t^\ell}{p_{t-1}^\ell} \qquad \forall \ell \in \mathscr{L}$$

**Aggregate wealth**: the sum of all individual wealth levels

$$W_t = \sum_{n=1}^{N} W_{n,t}$$

**Individual wealth shares**: the ratios of each individual wealth out of aggregate wealth

$$\varphi_{n,t} = \frac{W_{n,t}}{W_t} \qquad \forall n \in \mathscr{N}$$

**Market portfolio**: the wealth-weighted sum of individual portfolios

$$\mathbf{x}_t = \sum_{n=1}^{N} \mathbf{x}_{n,t} \cdot \varphi_{n,t}$$

# Trader optimisation problem

At every time step, each trader faces an optimisation problem of the form:

$$\max_{\mathbf{x}_t} \ \mathbb{E}[U(W_t)]$$

s.t.

$$W_t = W_{t-1} \cdot \left[ x_{t-1}^0 \cdot (1 + r_f) + \sum_{\ell=1}^{L} x_{t-1}^{\ell} \cdot \left( \frac{p_t^{\ell}}{p_{t-1}^{\ell}} + e_t^{\ell} \right) \right]$$

For notational convenience the *n* subscript is dropped here. Coherently with CRRA attitude, the utility function reads:

$$U(W_t) = \frac{W_t^{1-\gamma} - 1}{1 - \gamma}$$

where $\gamma > 0$ denotes the risk-aversion coefficient.

# Trader expectations

We assume the trader forms expectations about future price returns and their (co)variances as smooth functions of the EWMA estimators over the information set previously defined:

$$\hat{\rho}_t^{\ell} = \lambda \cdot \sum_{\tau=0}^{\infty} (1-\lambda)^{\tau} \cdot \rho_{t-\tau-1}^{\ell}$$

$$\hat{\sigma}_{\rho,t}^{\ell,h} = \lambda \cdot \sum_{\tau=0}^{\infty} (1-\lambda)^{\tau} \cdot \left[\rho_{t-\tau-1}^{\ell} - \hat{\rho}_{t-\tau-1}^{\ell}\right] \cdot \left[\rho_{t-\tau-1}^{h} - \hat{\rho}_{t-\tau-1}^{h}\right]$$

where $\rho_t^{\ell} = \frac{p_t^{\ell}}{p_{t-1}^{\ell}} - 1$ is the price return of security $\ell$ between $t-1$ and $t$. The decay factor $\lambda \in [0,1]$ captures the way relative weights are distributed across more recent and older observations.

# Trader investment function

We adopt the same mean-variance approximation of the optimal investment function proposed in Chiarella and He (2001):

$$x_t = f\left(\mathcal{I}_t\right) = \frac{1}{\gamma} \cdot \hat{\mathbb{C}}_t^{-1} \cdot \left[\, \mathbb{E}_t - r_f \cdot \mathbf{1} \right]$$

where $\mathbb{E}_t$ and $\hat{\mathbb{C}}_t^{-1}$ are, respectively, the vector of expected total returns and the inverse of the expected variance-covariance matrix, whose elements read:

$$\mathbb{E}_t^{\ell} = \overline{e}^{\ell} + d \cdot \hat{\rho}_t^{\ell}$$
$$\hat{\mathbb{C}}_t^{\ell,h} = \hat{\sigma}_{\rho,t}^{\ell,h} + \sigma_e^{\ell,h}$$

where $d$ is a behavioural parameter:

- $d = 0$ trader is a fundamentalist;
- $d > 0$ trader is a trend-chaser;
- $d < 0$ trader is a trend-contrarian.

# Demand and equilibrium condition

Individual demand for a risky asset reads:

$$Z_{n,t}^{\ell} = \frac{x_{n,t}^{\ell} \cdot W_{n,t}}{p_t^{\ell}} \qquad \forall \ell \in \mathscr{L}$$

Normalising the supply of each risky asset to 1, the equilibrium condition reads:

$$\sum_{n=1}^{N} \mathbf{Z}_{n,t} = \mathbf{1}$$

Solving for single security prices yields:

$$p_t^{\ell} = \sum_{n=1}^{N} x_{n,t}^{\ell} \cdot W_{n,t}(\mathbf{p}_t) \qquad \forall \ell \in \mathscr{L}$$

where prices appear both in the LHS and RHS of the equation, as argument of wealth evolution.

# Pricing of risky assets

## Proposition

*If short positions are not allowed, i.e.*

$$x_{n,t}^{\ell} \in (0,1) \qquad \forall n \in \mathscr{N},\ \forall \ell \in \mathscr{L},\ \forall t$$

*then prevailing prices exist, are unique and strictly positive. It holds:*

$$W_t = W_{t-1} \cdot \frac{x_{t-1}^{0} \cdot (1 + r_f) + \sum_{\ell=1}^{L} x_{t-1}^{\ell} \cdot e_t^{\ell}}{x_t^{0}}$$

$$p_t^{\ell} = p_{t-1}^{\ell} \cdot \frac{x_t^{\ell}}{x_{t-1}^{\ell}} \cdot \frac{x_{t-1}^{0} \cdot (1 + r_f) + \sum_{\ell=1}^{L} x_{t-1}^{\ell} \cdot e_t^{\ell}}{x_t^{0}}$$

## Proof.

See **Appendix A** in the paper. □

# Simulation results: survival patterns

Our first simulation studies the way market selection occurs among different trading strategies by means of individual wealth shares $\varphi_{n,t}$

**Definition**

A trader $n$ is said to *"survive"* the economy if his long-run wealth-share is significantly different from 0, i.e. if $\lim_{t \to \infty} \varphi_{n,t} > 0$.
A trader $n$ is said to *"dominate"* the economy if his long-run wealth-share is significantly close to 1, i.e. if $\lim_{t \to \infty} \varphi_{n,t} = 1$.

Following Anufriev et al. (2006), two types of equilibria are possible:

1. Single-survivor equilibria (most '*aggressive*' trader);
2. Multiple-survivor equilibria (though non-generic).

# Survival patterns (cont'd)

| Description | Value |
|---|---|
| Initial population size | $N = 200$ |
| Number of risky assets | $L = 1$ |
| Static population | `true` |
| Riskless rate of return | $r_f = 0.02$ |
| $\gamma$ distribution | $\gamma_n \sim \mathcal{U}(1.0, 1000.0)$ |
| $\lambda$ distribution | $\lambda_n = 0.1,\ \forall n \in \mathscr{N}$ |
| $d$ distribution | $d_n = 1.0,\ \forall n \in \mathscr{N}$ |
| Initial wealth endowment | $W_{n,0} = 50.0,\ \forall n \in \mathscr{N}$ |
| Yield mean | $\overline{e} = 0.04$ |
| Yield variance | $\sigma_e^2 = 1.0\text{e}{-4}$ |
| Yield realisation distribution | $e_t \sim \mathcal{N}(\overline{e}, \sigma_e^2)$ |
| Initial risky asset price level | $p_0 = 0.1$ |
| $x_n$ admissible interval | $x_{n,t} \in [0.01, 0.99],\ \forall n \in \mathscr{N}, \forall t$ |

Table: Parameters and initial conditions (1)

# Survival patterns (cont'd)

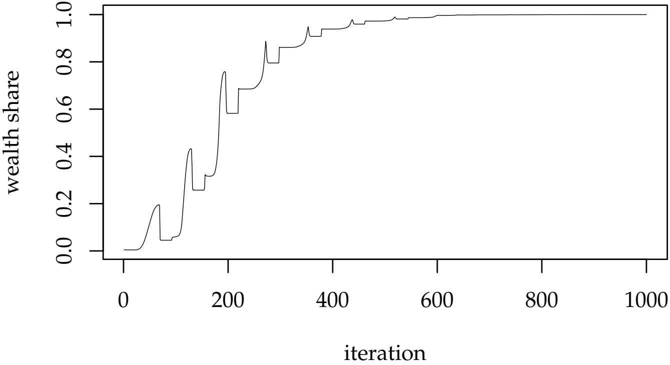



Figure: Evolution of wealth-share for the least-risk-averse trader. Single-survivor.

# Survival patterns (cont'd)

| Description | Value |
|---|---|
| Initial population size | $N = 200$ |
| Number of risky assets | $L = 1$ |
| Static population | `true` |
| Riskless rate of return | $r_f = 0.02$ |
| $\gamma$ distribution | $\gamma_n \sim \mathcal{U}(100.0, 1000.0)$ |
| $\lambda$ distribution | $\lambda_n = 0.1,\ \forall n \in \mathscr{N}$ |
| $d$ distribution | $d_n = 1.0,\ \forall n \in \mathscr{N}$ |
| Initial wealth endowment | $W_{n,0} = 50.0,\ \forall n \in \mathscr{N}$ |
| Yield mean | $\overline{e} = 0.04$ |
| Yield variance | $\sigma_e^2 = 1.0\text{e}{-4}$ |
| Yield realisation distribution | $e_t \sim \mathcal{N}(\overline{e}, \sigma_e^2)$ |
| Initial risky asset price level | $p_0 = 0.1$ |
| $x_n$ admissible interval | $x_{n,t} \in [0.01, 0.99],\ \forall n \in \mathscr{N}, \forall t$ |

Table: Parameters and initial conditions (2)

# Survival patterns (cont'd)

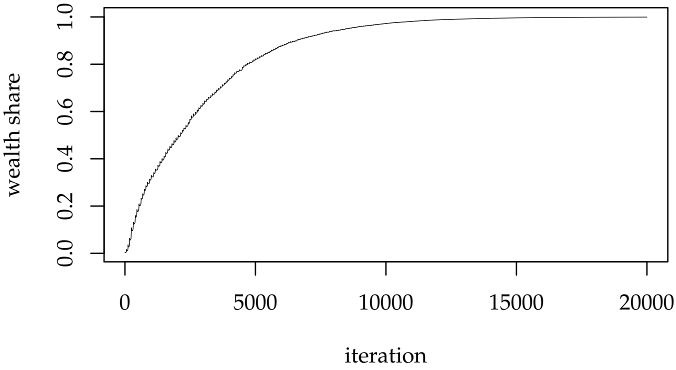


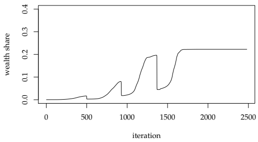

Figure: Evolution of wealth-share for the least-risk-averse trader. Single-survivor.

# Survival patterns (cont'd)

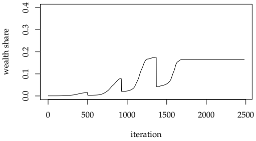
| Description | Value |
|---|---|
| Initial population size | $N = 200$ |
| Number of risky assets | $L = 1$ |
| Static population | `true` |
| Riskless rate of return | $r_f = 0.02$ |
| $\gamma$ distribution | $\gamma_n \sim \mathcal{U}(1.0, 1000.0)$ |
| $\lambda$ distribution | $\lambda_n = 0.01, \ \forall n \in \mathscr{N}$ |
| $d$ distribution | $d_n = 1.0, \ \forall n \in \mathscr{N}$ |
| Initial wealth endowment | $W_{n,0} = 50.0, \ \forall n \in \mathscr{N}$ |
| Yield mean | $\overline{e} = 0.04$ |
| Yield variance | $\sigma_e^2 = 1.0\text{e}{-4}$ |
| Yield realisation distribution | $e_t \sim \mathcal{N}(\overline{e}, \sigma_e^2)$ |
| Initial risky asset price level | $p_0 = 0.1$ |
| $x_n$ admissible interval | $x_{n,t} \in [0.01, 0.99], \ \forall n \in \mathscr{N}, \forall t$ |

Table: Parameters and initial conditions (3)

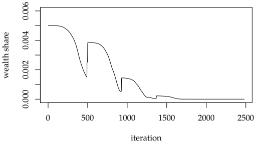
# Survival patterns (cont'd)




(a) lowest $\gamma_n$




(b) second-lowest $\gamma_n$




(c) highest $\gamma_n$

Figure: Multiple-survivor equilibrium. Evolution of wealth-shares.

# Survival patterns (cont'd)

Heterogeneity in the risk-aversion coefficient, within the stability domain of the system, triggers a wealth-driven selection mechanism.

- For a short memory-span (large $\lambda$), the least risk-averse trader survives and dominates the economy;
- For a long memory-span (small $\lambda$), multiple traders, still low-risk averse, survive and display identical investment decisions.

Out of the stability domain of the system (i.e. for large enough $\lambda$, following Anufriev et al. 2006) selection does not occur: individual wealth-shares keep fluctuating indefinitely with no clear-cut outcome.

# Simulation results: transitional price dynamics

| Description | Value |
|---|---|
| Initial population size | $N = 1,000$ |
| Number of risky assets | $L = 1$ |
| Static population | `true` |
| Riskless rate of return | $r_f = 0.02$ |
| $\gamma$ distribution | $\gamma_n \sim \mathcal{U}(1.0, 500.0)$ |
| $\lambda$ distribution | $\lambda_n = 0.0036,\ \forall n \in \mathscr{N}$ |
| $d$ distribution | $d_n = 1.0,\ \forall n \in \mathscr{N}$ |
| Initial wealth endowment | $W_{n,0} = 50.0,\ \forall n \in \mathscr{N}$ |
| Yield mean | $\overline{e} = 0.04$ |
| Yield variance | $\sigma_e^2 = 1.0\text{e}{-4}$ |
| Yield realisation distribution | $e_t \sim \mathcal{N}(\overline{e}, \sigma_e^2)$ |
| Initial risky asset price level | $p_0 = 0.1$ |
| $x_n$ admissible interval | $x_{n,t} \in [0.01, 0.99],\ \forall n \in \mathscr{N}, \forall t$ |

Table: Parameters and initial conditions (4)

# Transitional price dynamics (cont'd)

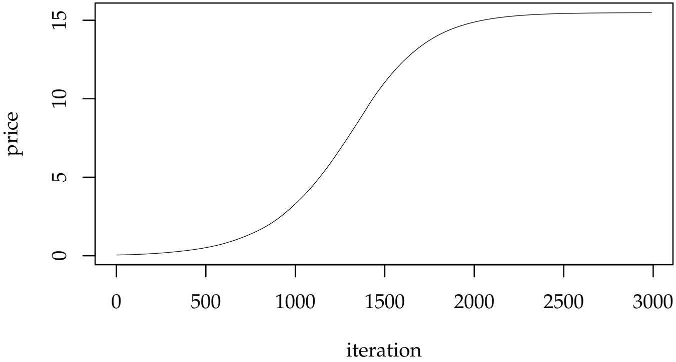



Figure: Price dynamics. Smooth and monotone convergence to equilibrium.

# Transitional price dynamics (cont'd)

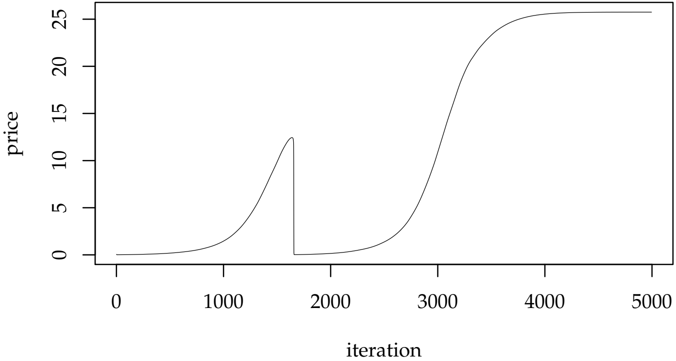



Figure: Price dynamics. Emergence of a bubble-and-bust cycle.
$\lambda_n = 0.00365, \ \forall n \in \mathcal{N}$

# Transitional price dynamics (cont'd)

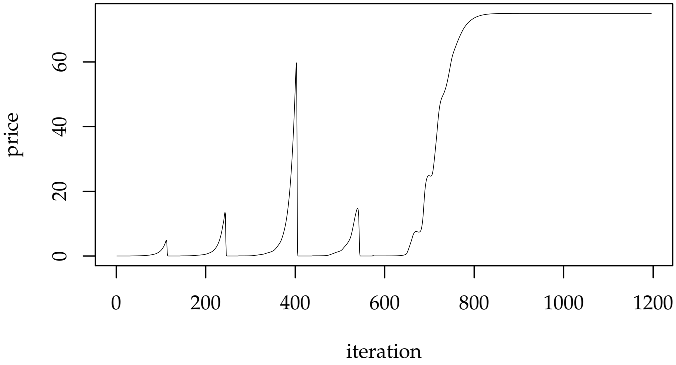



Figure: Price dynamics. Emergence of multiple bubble-and-bust cycles.
$\lambda_n = 0.155, \ \forall n \in \mathcal{N}$

# Transitional price dynamics (cont'd)

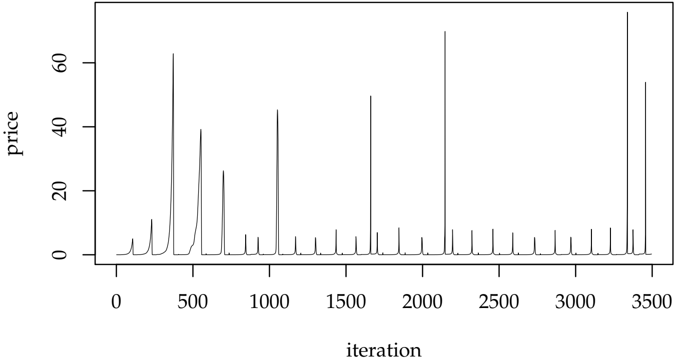



Figure: Price dynamics. No convergence to an equilibrium value.
$\lambda_n = 0.16, \ \forall n \in \mathcal{N}$

# Transitional price dynamics (cont'd)

The emergent properties observed in a trend-chasers-only setting maintain robustness with respect to the introduction of fundamentalist and trend-contrarian traders in the economy.

- Fundamentalists are expected to stabilise the price as they act against chartists whenever current price deviates from its fundamental value;
- Contrarians shall counteract the attempt made by trend-chasers to exacerbate the price trend by acting in a symmetrical fashion.

We differentiate the $d$ parameter in order to model a population largely composed of quasi-fundamentalists and well balanced crowds of trend-chasers and contrarians.

# Transitional price dynamics (cont'd)

We now shift the analysis to the transitional price dynamics:

| Description | Value |
|---|---|
| Initial population size | $N = 1,000$ |
| Number of risky assets | $L = 1$ |
| Static population | `true` |
| Riskless rate of return | $r_f = 0.02$ |
| $\gamma$ distribution | $\gamma_n \sim \mathcal{U}(1.0, 500.0)$ |
| $\lambda$ distribution | $\lambda_n = 0.1, \ \forall n \in \mathscr{N}$ |
| $d$ distribution | $d_n \sim \mathcal{N}(0, 1)$ |
| Initial wealth endowment | $W_{n,0} = 50.0, \ \forall n \in \mathscr{N}$ |
| Yield mean | $\overline{e} = 0.04$ |
| Yield variance | $\sigma_e^2 = 1.0\text{e}{-4}$ |
| Yield realisation distribution | $e_t \sim \mathcal{N}(\overline{e}, \sigma_e^2)$ |
| Initial risky asset price level | $p_0 = 0.1$ |
| $x_n$ admissible interval | $x_{n,t} \in [0.01, 0.99], \ \forall n \in \mathscr{N}, \forall t$ |

Table: Parameters and initial conditions (5)

# Transitional price dynamics (cont'd)

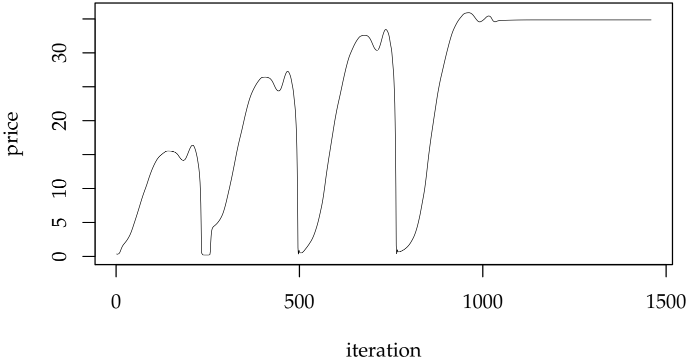



Figure: Price dynamics. Fundamentalists vs. chartists.

# Transitional price dynamics (cont'd)

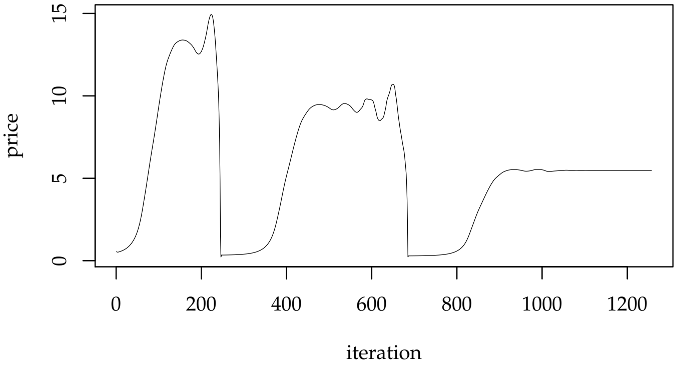



Figure: Price dynamics. Fundamentalists vs. chartists. $\gamma_n \sim \mathcal{U}(1.0, 1000.0)$

# Concluding remarks

1. Analysis of individual wealth-shares
   - strong market selection mechanism
   - single- and multiple-survivor equilibria
   - riskier investment functions globally dominate
2. Analysis of transitional price dynamics
   - emergence of bubble-and-bust cycles
   - robust to the introduction of fundamentalist and trend-contrarian

**In a nutshell**

risk-aversion heterogeneity

$\Downarrow$

decoupling of price dynamics from the fundamental yield process

# Conceivable improvements

Our framework can be extended in a number of directions:

- multiple risky assets;
- dynamic population;
- more realistic traders' behaviour (prospect theory, herding) and learning (genetic algorithms, classifier systems).

Our conjecture is that a sharper departure from rationality assumption is needed in order to obtain dynamics that are closer to reality.

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