Bubble-and-bust dynamics under walrasian asset pricing and heterogeneous traders

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Outline

1. Empirical stylised facts of financial time-series
2. Methodological perspective
3. The model
4. Simulation and results
5. Concluding remarks and conceivable extension
Empirical stylised facts

Empirical analysis of actual financial data often yields puzzling statistical properties. Notably, asset returns time-series use to exhibit:

- Volatility clustering;
- Excess volatility;
- Excess covariance;
- Heavy tails.

To some extent at odds with accepted theory and practice.
Financial bubbles: a largely neglected stylised fact

Source: NASDAQ OMX Group

Shaded areas indicate US recessions - 2014 research.stlouisfed.org
“if the reason that the price is high today is only because investors believe that the selling price will be high tomorrow – when ‘fundamental’ factors do not seem to justify such a price – then a bubble exists” (Stiglitz)

“a sharp rise in the price of an asset or a range of assets in a continuous process, with the initial rise generating expectations of further rises and attracting new buyers – generally speculators interested in profits from trading in the asset rather than its use or earnings capacity” (Kindleberger)
Our proposal

We set up a model able to yield:

- **endogenous** bubble-and-bust dynamics
- as a result of the sole interaction among *heterogeneous adaptive traders*
- highlighting booms and crashes as intrinsic features of financial markets.
Methodological perspective

### Heterogeneous Agents Models

- analytical investigations of the dynamical systems representing the laws of motion of the economy;
- analytical tractability often leads to restrictive simplifying assumptions;
- focus on asymptotic properties.

### Agent-Based Models

- computational (numerical) study of economies modelled as evolving systems of interacting agents;
- complex behaviour specifications;
- keep track of the whole dynamics.
The model

Following Anufriev et al. (2012), consider a pure-exchange economy:

- \( N \) heterogeneous traders (index \( \mathcal{N} = \{1, \ldots, n, \ldots, N\} \));
- \( L \) long-lived risky securities (index \( \mathcal{L} = \{1, \ldots, \ell, \ldots, L\} \));
- a riskless bond;
- time is discrete;
- risky securities, present in fixed amount, have ex-dividend price \( p^\ell_t \) and pay a random dividend \( d^\ell_t \) at the end of each period;
- the bond, inelastically supplied, have price normalized to 1 (numéraire) and yields \( r_f > 0 \) in every \( t \);
- trader wealth equals the market value of the portfolio he holds:

\[
W_{n,t} = \sum_{\ell=1}^{L} A^\ell_{n,t} \cdot p^\ell_t + B_{n,t}
\]
Trader behaviour

At the beginning of each time step, trader $n$ invests a share $x_{n,t}^\ell$ of his wealth in security $\ell$; the decision is made according to the information set

$$\mathcal{I}_t = \{p_1^{\tau}, \ldots, p_L^{\tau}; \ d_1^{\tau}, \ldots, d_L^{\tau} \mid \tau < t\}$$

that is common knowledge, and to trader-specific investment function

$$f_n : \mathbb{R}^{\tau \times L} \rightarrow \mathbb{R}^L \text{ such that } x_{n,t} = f_{n,t}(\mathcal{I}_t)$$

that is independent on wealth, as we assume overall CRRA attitude. The amount of wealth invested in the bond is residually determined:

$$x_{n,t}^0 = 1 - \sum_{\ell=1}^L x_{n,t}^\ell$$

The evolution of individual wealth, in terms of wealth fractions, holds:

$$W_{n,t} = W_{n,t-1} \cdot \left[ x_{n,t-1}^0 \cdot (1 + r_f) + \sum_{\ell=1}^L x_{n,t-1}^\ell \cdot \left( \frac{p_t^\ell}{p_{t-1}^\ell} + e_t^\ell \right) \right]$$
Useful additional definitions

**Dividend yield**: the ratio of dividend over past realized price (proxy for fundamentals)

\[ e_t^\ell = \frac{d_t^\ell}{p_{t-1}^\ell} \quad \forall \ell \in \mathcal{L} \]

**Aggregate wealth**: the sum of all individual wealth levels

\[ W_t = \sum_{n=1}^{N} W_{n,t} \]

**Individual wealth shares**: the ratios of each individual wealth out of aggregate wealth

\[ \varphi_{n,t} = \frac{W_{n,t}}{W_t} \quad \forall n \in \mathcal{N} \]

**Market portfolio**: the wealth-weighted sum of individual portfolios

\[ x_t = \sum_{n=1}^{N} x_{n,t} \cdot \varphi_{n,t} \]
Trader optimisation problem

At every time step, each trader faces an optimisation problem of the form:

\[
\max_{x_t} \mathbb{E}[U(W_t)]
\]

s.t.

\[
W_t = W_{t-1} \cdot \left[ x_{t-1}^0 \cdot (1 + r_f) + \sum_{\ell=1}^{L} x_{t-1}^\ell \cdot \left( \frac{p^\ell_t}{p^\ell_{t-1}} + e^\ell_t \right) \right]
\]

For notational convenience the \( n \) subscript is dropped here. Coherently with CRRA attitude, the utility function reads:

\[
U(W_t) = \frac{W_t^{1-\gamma} - 1}{1 - \gamma}
\]

where \( \gamma > 0 \) denotes the risk-aversion coefficient.
Trader expectations

We assume the trader forms expectations about future price returns and their (co)variances as smooth functions of the EWMA estimators over the information set previously defined:

$$\hat{\rho}_t^\ell = \lambda \cdot \sum_{\tau=0}^{\infty} (1 - \lambda)^\tau \cdot \rho_{t-\tau-1}^\ell$$

$$\hat{\sigma}_{\rho,t}^{\ell,h} = \lambda \cdot \sum_{\tau=0}^{\infty} (1 - \lambda)^\tau \cdot \left[ \rho_{t-\tau-1}^\ell - \hat{\rho}_{t-\tau-1}^\ell \right] \cdot \left[ \rho_{t-\tau-1}^h - \hat{\rho}_{t-\tau-1}^h \right]$$

where $\rho_t^\ell = \frac{p_t^\ell}{p_{t-1}^\ell} - 1$ is the price return of security $\ell$ between $t - 1$ and $t$. The decay factor $\lambda \in [0, 1]$ captures the way relative weights are distributed across more recent and older observations.
We adopt the same mean-variance approximation of the optimal investment function proposed in Chiarella and He (2001):

\[
x_t = f(I_t) = \frac{1}{\gamma} \cdot \hat{C}_t^{-1} \cdot \left[ E_t - r_f \cdot 1 \right]
\]

where \( E_t \) and \( \hat{C}_t^{-1} \) are, respectively, the vector of expected total returns and the inverse of the expected variance-covariance matrix, whose elements read:

\[
E^\ell_t = \bar{e}^\ell + d \cdot \hat{\rho}^\ell_t
\]

\[
\hat{C}^{\ell,h}_t = \hat{\sigma}^{\ell,h}_{\rho,t} + \sigma^{\ell,h}_e
\]

where \( d \) is a behavioural parameter:

- \( d = 0 \) trader is a fundamentalist;
- \( d > 0 \) trader is a trend-chaser;
- \( d < 0 \) trader is a trend-contrarian.
Individual demand for a risky asset reads:

$$Z_{n,t}^\ell = \frac{x_{n,t}^\ell \cdot W_{n,t}}{p_t^\ell} \quad \forall \ell \in \mathcal{L}$$

Normalising the supply of each risky asset to 1, the equilibrium condition reads:

$$\sum_{n=1}^{N} Z_{n,t} = 1$$

Solving for single security prices yields:

$$p_t^\ell = \sum_{n=1}^{N} x_{n,t}^\ell \cdot W_{n,t}(p_t) \quad \forall \ell \in \mathcal{L}$$

where prices appear both in the LHS and RHS of the equation, as argument of wealth evolution.
Proposition

If short positions are not allowed, i.e.

\[ x_{n,t}^\ell \in (0, 1) \quad \forall n \in N, \quad \forall \ell \in L, \quad \forall t \]

then prevailing prices exist, are unique and strictly positive. It holds:

\[
W_t = W_{t-1} \cdot \frac{x_{t-1}^0 \cdot (1 + r_f) + \sum_{\ell=1}^L x_{t-1}^\ell \cdot e_t^\ell}{x_t^0} \\
p_t^\ell = p_{t-1}^\ell \cdot \frac{x_t^\ell}{x_{t-1}^\ell} \cdot \frac{x_{t-1}^0 \cdot (1 + r_f) + \sum_{\ell=1}^L x_{t-1}^\ell \cdot e_t^\ell}{x_t^0}
\]

Proof.

See Appendix A in the paper.
Simulation results: survival patterns

Our first simulation studies the way market selection occurs among different trading strategies by means of individual wealth shares $\varphi_{n,t}$

**Definition**

A trader $n$ is said to “survive” the economy if his long-run wealth-share is significantly different from 0, i.e. if $\lim_{t \to \infty} \varphi_{n,t} > 0$.

A trader $n$ is said to “dominate” the economy if his long-run wealth-share is significantly close to 1, i.e. if $\lim_{t \to \infty} \varphi_{n,t} = 1$.

Following Anufriev et al. (2006), two types of equilibria are possible:

1. Single-survivor equilibria (most ‘aggressive’ trader);
2. Multiple-survivor equilibria (though non-generic).
### Survival patterns (cont’d)

<table>
<thead>
<tr>
<th>Description</th>
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</tr>
</thead>
<tbody>
<tr>
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<td>$N = 200$</td>
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<tr>
<td>Number of risky assets</td>
<td>$L = 1$</td>
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<tr>
<td>Static population</td>
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<tr>
<td>Riskless rate of return</td>
<td>$r_f = 0.02$</td>
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<td>$\gamma$ distribution</td>
<td>$\gamma_n \sim \mathcal{U}(1.0, 1000.0)$</td>
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<td>$\overline{e} = 0.04$</td>
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<td>Yield realisation distribution</td>
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**Table:** Parameters and initial conditions (1)
Figure: Evolution of wealth-share for the least-risk-averse trader. Single-survivor.
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**Table:** Parameters and initial conditions (2)
Figure: Evolution of wealth-share for the least-risk-averse trader. Single-survivor.
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**Table:** Parameters and initial conditions (3)
Survival patterns (cont’d)

Figure: Multiple-survivor equilibrium. Evolution of wealth-shares.

(a) lowest $\gamma_n$  

(b) second-lowest $\gamma_n$  

(c) highest $\gamma_n$
Heterogeneity in the risk-aversion coefficient, within the stability domain of the system, triggers a wealth-driven selection mechanism.

- For a short memory-span (large $\lambda$), the least risk-averse trader survives and dominates the economy;
- For a long memory-span (small $\lambda$), multiple traders, still low-risk averse, survive and display identical investment decisions.

Out of the stability domain of the system (i.e. for large enough $\lambda$, following Anufriev et al. 2006) selection does not occur: individual wealth-shares keep fluctuating indefinitely with no clear-cut outcome.
## Simulation results: transitional price dynamics

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**Table:** Parameters and initial conditions (4)
Figure: Price dynamics. Smooth and monotone convergence to equilibrium.
Transitional price dynamics (cont’d)

Figure: Price dynamics. Emergence of a bubble-and-bust cycle. \( \lambda_n = 0.00365, \ \forall n \in \mathcal{N} \)
Figure: Price dynamics. Emergence of multiple bubble-and-bust cycles. 
\[ \lambda_n = 0.155, \quad \forall n \in \mathcal{N} \]
Figure: Price dynamics. No convergence to an equilibrium value. 
\[ \lambda_n = 0.16, \quad \forall n \in \mathcal{N} \]
The emergent properties observed in a trend-chasers-only setting maintain robustness with respect to the introduction of fundamentalist and trend-contrarian traders in the economy.

- Fundamentalists are expected to stabilise the price as they act against chartists whenever current price deviates from its fundamental value;
- Contrarians shall counteract the attempt made by trend-chasers to exacerbate the price trend by acting in a symmetrical fashion.

We differentiate the $d$ parameter in order to model a population largely composed of quasi-fundamentalists and well balanced crowds of trend-chasers and contrarians.
We now shift the analysis to the transitional price dynamics:

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Table: Parameters and initial conditions (5)
Figure: Price dynamics. Fundamentalists vs. chartists.
Transitional price dynamics (cont’d)

Figure: Price dynamics. Fundamentalists vs. chartists. $\gamma_n \sim U(1.0, 1000.0)$
Concluding remarks

1. Analysis of individual wealth-shares
   - strong market selection mechanism
   - single- and multiple-survivor equilibria
   - riskier investment functions globally dominate

2. Analysis of transitional price dynamics
   - emergence of bubble-and-bust cycles
   - robust to the introduction of fundamentalist and trend-contrarian

In a nutshell

risk-aversion heterogeneity

\[\Downarrow\]

decoupling of price dynamics from the fundamental yield process
Conceivable improvements

Our framework can be extended in a number of directions:

- multiple risky assets;
- dynamic population;
- more realistic traders’ behaviour (prospect theory, herding) and learning (genetic algorithms, classifier systems).

Our conjecture is that a sharper departure from rationality assumption is needed in order to obtain dynamics that are closer to reality.
Selected references


