# Exercises on Cooperative Games

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## Excercise 1

*Construct a convex 3-player game; compute the marginal contribution vectors, draw its core in the simplex, compute the (symmetric) Shapley value and the Shapley value associated to non-uniform weights (of your choice).*

The game is a couple $\langle N, v \rangle$, with players $N = \{1, 2, 3\}$, so that $n = 3$; characteristic function $v : S \subseteq N \longrightarrow v(S) \in \mathbb{R}$, where $S$'s denote the $2^3 = 8$ possible coalitions including $\varnothing$, reads:

$$v(\varnothing) \stackrel{\text{def}}{=} 0$$

$$v(i) = 1 \quad \forall i = 1, 2, 3$$

$$v(1, 2) = 3$$

$$v(1, 3) = 4$$

$$v(2, 3) = 5$$

$$v(N) = 10$$

It is immediate to check for convexity, as is sufficient to prove that the marginal contribution $MC_i(S)$ of each player $i$ to a coalition $S \subseteq N, i \in S$ he participates to is non-decreasing over the size $|S|$ of the coalition itself. See table 1.

Denoting by $\Pi_3$ the set of all $n! = 3! = 6$ possible permutations $\pi_i$ of players and computing the marginal contribution of each player for each progressively enlarging coalition, yield the marginal contribution vectors. See table 2 on the next page (neglect the last row for a while).

| $S_i \subset N$ | $\lvert S_i \rvert$ | $MC_1(S_i)$ | $MC_2(S_i)$ | $MC_3(S_i)$ |
|---|---|---|---|---|
| $S_1 = \{1\}$ | 1 | 1 | $*$ | $*$ |
| $S_3 = \{2\}$ | 1 | $*$ | 1 | $*$ |
| $S_3 = \{3\}$ | 1 | $*$ | $*$ | 1 |
| $S_4 = \{1, 2\}$ | 2 | 2 | 2 | $*$ |
| $S_5 = \{1, 3\}$ | 2 | 3 | $*$ | 3 |
| $S_6 = \{2, 3\}$ | 2 | $*$ | 4 | 4 |
| $S_7 = \{1, 2, 3\}$ | 3 | 5 | 6 | 7 |

Table 1: Marginal contribution of each player to all coalition he belongs to.

| $\pi_i \in \Pi_3$ | $\mu_1(\pi_i)$ | $\mu_2(\pi_i)$ | $\mu_3(\pi_i)$ | $v(N)$ |
|---|---|---|---|---|
| $\pi_1 = \{1,2,3\}$ | 1 | 2 | 7 | 10 |
| $\pi_2 = \{1,3,2\}$ | 1 | 6 | 3 | 10 |
| $\pi_3 = \{2,1,3\}$ | 2 | 1 | 7 | 10 |
| $\pi_4 = \{2,3,1\}$ | 5 | 1 | 4 | 10 |
| $\pi_5 = \{3,1,2\}$ | 3 | 6 | 1 | 10 |
| $\pi_6 = \{3,2,1\}$ | 5 | 4 | 1 | 10 |
| $\sum_{\pi \in \Pi_3} \mu_i(\pi)$ | 17 | 20 | 23 | * |

Table 2: Permutations and marginal contribution vectors.

Looking at the three central columns, Marginal contribution vectors read:

$\mu(\pi_1) = (1,2,7)$

$\mu(\pi_2) = (1,6,3)$

$\mu(\pi_3) = (2,1,7)$

$\mu(\pi_4) = (5,1,4)$

$\mu(\pi_5) = (3,6,1)$

$\mu(\pi_6) = (5,4,1)$

In this case all marginal contribution vectors are distinct. Due to convexity of the ($TU$) game, the convex hull of these vectors (i.e. the Weber set) identifies the core, i.e. the set of allocations that are socially stable:

$$C(N,v) = \{x \in \mathbb{R}^3 | \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S), \forall S \subset N\}$$

The latter is, in turns, a subset of the set of imputations satifying both individual and collective rationality, defined as:

$$I(N,v) = \{x \in \mathbb{R}^3 | \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(i), \forall i \in N\}$$

This set corresponds to a convex polyhedron in the regular simplex, bounded by the lines representing the worth of single-player coalitions (the non-cooperative equivalent):

$$\sum_{i \in N} x_i = v(N) \quad \xrightarrow{implies} \quad x_1 + x_2 + x_3 = 10$$

$$x_i \geq v(i), \ \forall i \in N \quad \xrightarrow{implies} \quad x_1 \geq 1, \quad x_2 \geq 1, \quad x_3 \geq 1$$

See figure 1 on the following page, representing both the set of imputations and the core of the game.

Alternatively, the core can be obtained by imposing to the set of imputations the additional constraint $\sum_{i \in S} x_i \geq v(S), \forall S \subset N$ (see figure 2 on the next page). It holds:

$$x_1 + x_2 \geq v(1,2) = 3 \qquad x_1 + x_3 \geq v(1,3) = 4 \qquad x_2 + x_3 \geq v(2,3) = 5$$

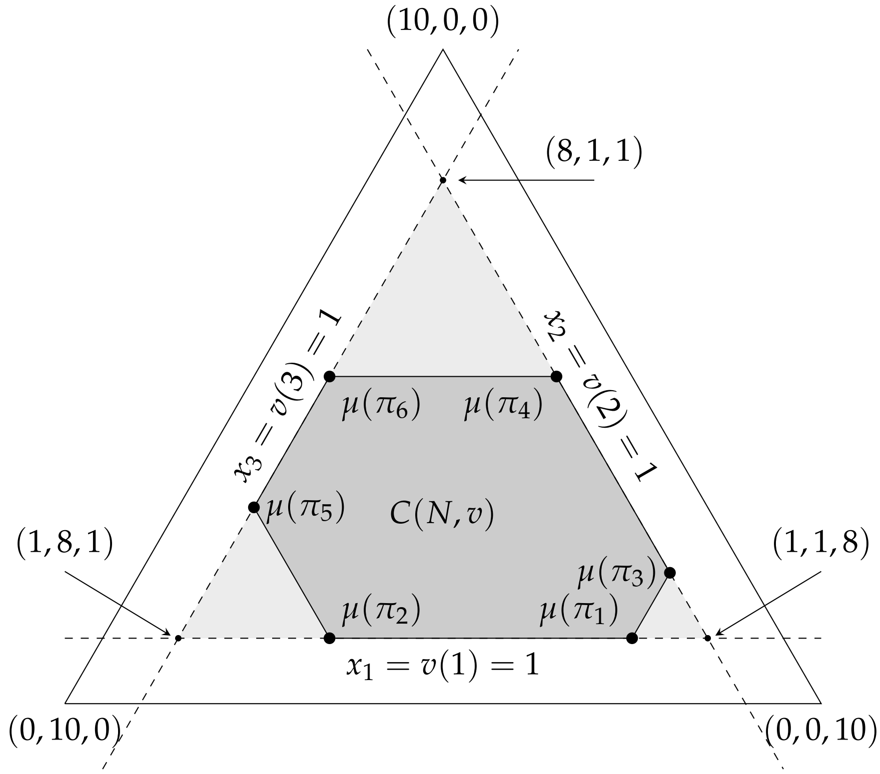


Figure 1: Set of imputations (light shaded) and core (dark shaded).

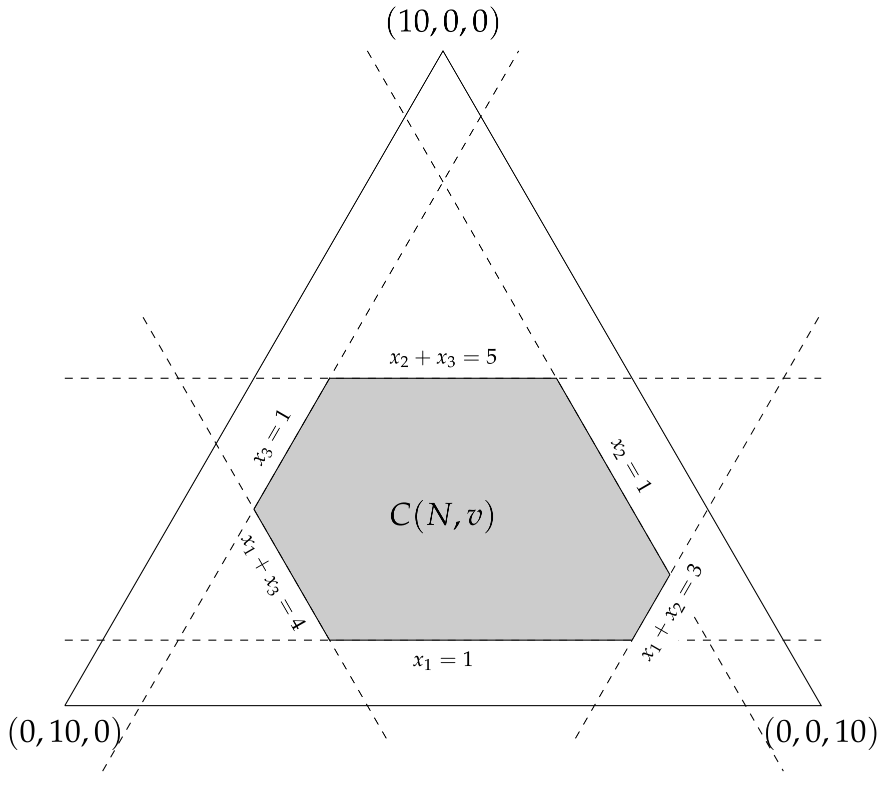


Figure 2: The core; alternative derivation.

The Shapley Value of the game, $SV(N,v)$, is an allocation $x \in \mathbb{R}^3$, $\sum_{i=1}^{3} x_i = 10$ that assigns to each player $i$ his average marginal contribution:

$$SV_i(N,v) \stackrel{\text{def}}{=} \frac{1}{n!} \sum_{\pi \in \Pi_n} \mu_i(\pi) \tag{$\clubsuit$}$$

Since the characteristic function is superadditive (as long as convexity implies it) this allocation satisfies, along with EFFiciency, SYMmetry, NULL player and ADDitivity, the axiom of individual rationality; it therefore belongs to the set of imputations. Moreover, thanks once again to convexity, the Shapley Value is a core allocation: it is a convex combination of marginal contribution vectors (see eq. $\clubsuit$). Moreover, since these latter are distinct, namely they all have the same multiplicity, the Value coincides with the average of core's vertices, $AV(N,v)$. It holds, therefore:

$$SV(N,v) = AV(N,v) \in C(N,v) \subset I(N,v)$$

$SV(N,v)$ can be easily computed by referring again to table 2 on page 2, now considering also the last row:

$$SV_1 = \frac{1}{6} \cdot 17 = \frac{17}{6} \qquad SV_2 = \frac{1}{6} \cdot 20 = \frac{10}{3} \qquad SV_3 = \frac{1}{6} \cdot 23 = \frac{23}{6}$$

Hence:

$$SV(N,v) = (2.8\overline{3}, 3.\overline{3}, 3.8\overline{3})$$

As expected the Shapley Value allocates a greater quantity to player 3 whose marginal contribution is (weakly) dominant in each class of coalitions by size (see table 1 on page 1), a smaller amount to player 2 (dominated by 3), and even smaller to player 1 (dominated by both). See figure 3 on the following page (some of previous notation is dropped). We can notice that the $SV$ allocation lies all but far from the center of gravity of the simplex, denoting some degree of intrinsic *fairness*. The above situation can be reversed by changing the weights the Value is computed with, thus turning from an uniform probability distribution $\lambda(\pi_i) = \frac{1}{6}, \forall i = 1, \ldots, 6$ and $\lambda(\pi) \in \Delta(\Pi_3)$ over permutations $\pi_i$, being $\Delta(\Pi_3) \subset \mathbb{R}^{n!}$ the simplex of probability distributions over $\Pi_3$, to a non-uniform one, say $\lambda'(\pi)$; suppose, for whatever reason, that permutations $\pi_i$ are no longer equally likely; for instance, player 1 is expected to join the coalition as the very last or, less likely, as second; in this case weights may read:

- ◇ $1/20$ for those permutations where player 1 is the first $(\pi_1, \pi_2)$;
- ◇ $1/5$ for those permutations where player 1 joins as second $(\pi_3, \pi_5)$;
- ◇ $1/4$ for those permutations where player 1 joins as third $(\pi_4, \pi_6)$;

determining the distribution:

$$\lambda'(\pi) = (0.05, 0.05, 0.2, 0.25, 0.2, 0.25)$$

Shapley Value computation then turns to:

$$WSV_i(N,v) = \frac{\mu_i(\pi_1) + \mu_i(\pi_2)}{20} + \frac{\mu_i(\pi_3) + \mu_i(\pi_5)}{5} + \frac{\mu_i(\pi_4) + \mu_i(\pi_6)}{4}$$

The Weighted Shapley Value, $WSV = (3.6, 3.05, 3.35)$, is again a core allocation, that now assigns a greater amount to player 1, although he is the least marginal contributor to any class of coalitions by size (see again figure 3 on the next page).

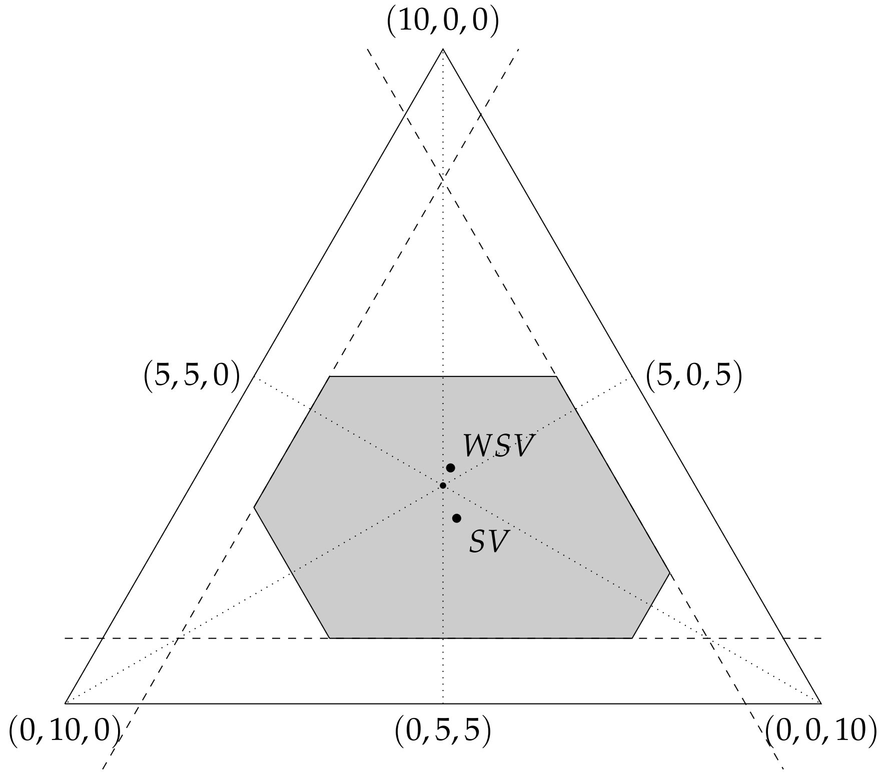


Figure 3: Position of the Shapley Value; center of gravity is also highlighted.

# Excercise 2

*Construct a 5 player decision game by listing the winning coalitions. Compute and compare the Banzhaf and Shapley-Shubik indices.*

The game is a couple $\langle N, W \rangle$, with players $N = \{1, 2, 3, 4, 5\}$, so that $n = 5$; the set of winning coalitions reads:

$$W = \{\{1, 2, 4\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{1, 2, 3, 4, 5\}\}$$

It is a weighted majority game, i.e. there exist a weight vector $w \in \mathbb{R}^3_+$ and a quota $Q \in \mathbb{R}_+$ such that:

$$S \in W \quad \text{if and only if} \quad \sum_{i \in S} w_i \geq Q$$

Possible values of $w$ and $Q$ read:

$$w = (1, 2, 0.5, 2, 0.5), \qquad Q = 5$$

Bearing identical weights in the (weighted majority) game, players 3 and 5 are substitutable, as well as 2 and 4. These latter, belonging to all winning coalitions, are, furthermore, veto players. The set of minimal winning coalitions $M$ consists of winning coalitions (of any size) whose members are all decisive:

$$M = \{S \in W \mid S \setminus i \notin W, \forall i \in S\} = \{\{1, 2, 4\}, \{2, 3, 4, 5\}\}$$

The set of smallest winning coalitions $W_S$ consists of a single coalition, whose size is 3 and all members are decisive; it is a subset of $M$:

$$W_S = \{\{1,2,4\}\} \subset M$$

The set of largest loosing coalitions $L$ consists of all 4-player coalitions excluding veto players or, equivalently, all 4-player coalitions not in $W$:

$$L = \{S \subset N \mid |S| = 4 \text{ and } \{2,4\} \not\subset S\} = \{\{1,2,3,5\}, \{1,3,4,5\}\}$$

The game satisfies the following properties:

- ◇ it is *essential*, as there is no dictator:

$$\nexists i \in N \quad \text{such that} \quad S \in W, \quad \forall S \subseteq N, i \in S$$

- ◇ the *grand coalition* is a winning coalition:

$$N = \{1,2,3,4,5\} \in W$$

- ◇ since there are veto players, there exists no couple of disjoint winning coalitions:

$$\forall S, T \in W, \qquad S \cap T \neq \varnothing$$

- ◇ enlarging a winning coalition keeps it winning:

$$S \in W, \quad T \subseteq N, \quad \longrightarrow \quad S \cup T \in W$$

The associated *simple game* $\langle N, v \rangle$ entails characteristic function of the form:

$$v(S) = \begin{cases} 1 & \text{if } S \in W \\ 0 & \text{if } S \notin W \end{cases}$$

This is useful to pinpoint decisive players, as it necessarily hold:

$$i \in S \text{ is decisive } \textit{wrt} \text{ coalition } S \text{ if and only if } v(S) - v(S \setminus i) = 1$$

i.e. a player is decisive in a coalition if and only if its marginal contribution to that coalition equals unity.
Denoting by $\mathscr{D}_i$ the set of all coalitions in which player $i$ is decisive, it holds:

$$\mathscr{D}_1 = \{\{1,2,4\}, \{1,2,3,4\}, \{1,2,4,5\}\}$$

$$\mathscr{D}_2 = W$$

$$\mathscr{D}_3 = \{\{2,3,4,5\}\}$$

$$\mathscr{D}_4 = W$$

$$\mathscr{D}_5 = \{\{2,3,4,5\}\}$$

The Banzhaf power Index, based on the number of coalitions in which a player is decisive, is therefore relatively easy to compute:

- ⋄ in its *raw* form:

$$\beta_i(N,v) = \sum_{S \subseteq N} [v(S) - v(S \setminus i)] = |\mathscr{D}_i| \quad \longrightarrow \quad \beta = (3,5,1,5,1)$$

- ⋄ in its proper version:

$$BI_i(N,v) = \frac{\beta_i(N,v)}{2^{n-1}} \quad \longrightarrow \quad BI = \left(\frac{3}{16}, \frac{5}{16}, \frac{1}{16}, \frac{5}{16}, \frac{1}{16}\right)$$

- ⋄ in its *normalized* form:

$$NBI_i(N,v) = \frac{\beta_i(N,v)}{\sum_{i \in N} \beta_i(N,v)} \quad \longrightarrow \quad NBI = \left(\frac{1}{5}, \frac{1}{3}, \frac{1}{15}, \frac{1}{3}, \frac{1}{15}\right)$$

Provided each $\mathscr{D}_i$ above, define $\mathscr{D}_{ij} \subseteq \mathscr{D}_i$ with $j = 1, \ldots, 5$ the subset of $j$-sized coalitions in which player $i$ is decisive; recalling the Shapley weight vector for $n = 5$, $\alpha_5 = \left(\frac{1}{5}, \frac{1}{20}, \frac{1}{30}, \frac{1}{20}, \frac{1}{5}\right)$, the Shapley-Shubik power Index $SSI$ reads:

$$SSI_i = \sum_{j=1}^{5} |\mathscr{D}_{ij}| \cdot \alpha_5(j) \quad \longrightarrow \quad SSI = \left(\frac{2}{15}, \frac{23}{60}, \frac{1}{20}, \frac{23}{60}, \frac{1}{20}\right)$$

In percentage points, it holds:

$$NBI = (20, 33.\overline{3}, 6.\overline{6}, 33.\overline{3}, 6.\overline{6})\% \qquad SSI = (13.\overline{3}, 38.\overline{3}, 5, 38.\overline{3}, 5)\%$$

As desirable both indices satisfy both efficiency (they sum up to unity) and symmetry (substitutable players bear the same power) axioms. It is immediate to notice that the image of $SSI$ is greater than that of $NBI$, so that extreme values in the second are made even more extreme in the first: for instance, veto players (2 and 4) who bear the maximum power, range from $33.\overline{3}\%$ under Banzhaf to $38.\overline{3}\%$ under Shapley and Shubik; analogously, players with least power range from $NBI_3 = NBI_5 = 6.\overline{6}\%$ to $SSI_3 = SSI_5 = 5\%$. Another remarkable difference, lies in the treatment of player 1: in particular, Banzhaf attributes him 50% more the power Shapley and Shubik do. The reason lies in fact these latter weight out the number of coalitions the player is decisive for, by the probability (depending on size) that such a coalition forms; Banzhaf, instead, just ponder that number over the raw sum of occasions any player is decisive in any possible coalition (independently of size).